A Variant of the Kochen-Specker Theorem Locating Value Indefiniteness

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Joint work with C. S. Calude and K. Svozil

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Between Kochen-Specker and Quantum Indeterminism

Kochen-Specker Theorem

- Impossibility of a (complete) noncontextual hidden variable theory for states in $d \geq 3$ Hilbert space
- "Quantum contextuality"
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There is a gap between this and the common interpretation of quantum measurements

Eigenvalue-Eigenstate link

A system in a state $|\psi\rangle$ has a definite property of an observable A if and only if $|\psi\rangle$ is an eigenstate of A.

Can we formally close this gap and show the *extent* of value indefiniteness?

The Kochen-Specker Theorem

A *context* in $\mathbb{C}^n$ is a set of n compatible (commuting) observables.

In $n \geq 3$ Hilbert space there is a finite set of (one-dimensional projection) observables $\mathcal{O}$ such that

the following three are in contradiction:

1. Every observable is assigned a definite value of 0 or 1;
2. These definite values are noncontextual;
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Greechie Diagrams & Value Assignments

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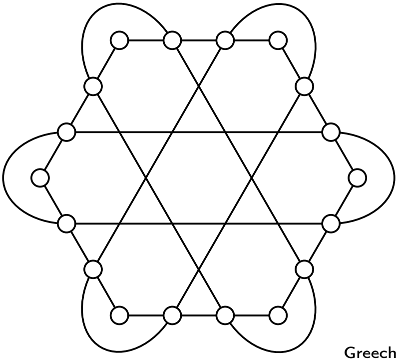

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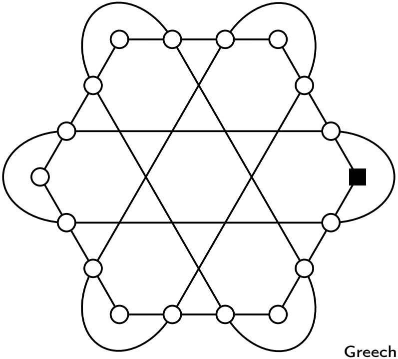

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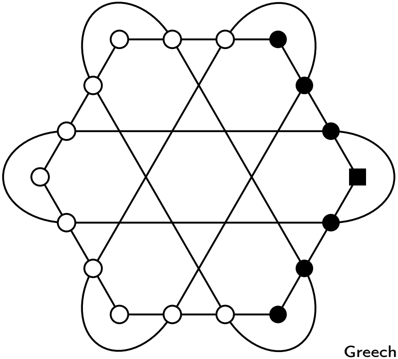

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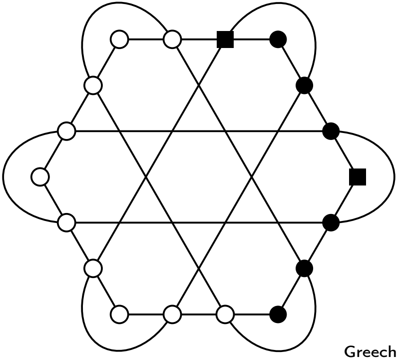

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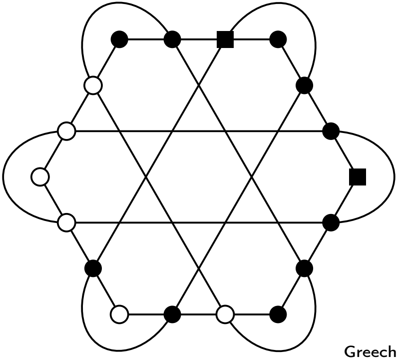

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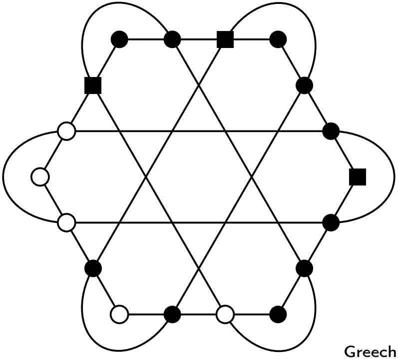

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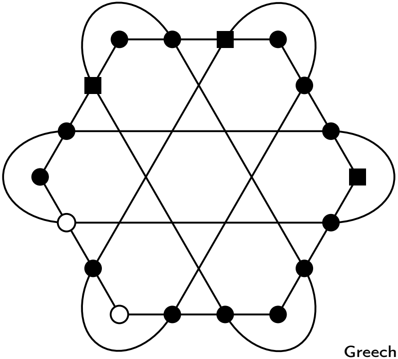

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The Extent of Value Indefiniteness

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Either, we reject:

- QM: But then we depart from quantum theory;
- NC: Definite values depends on measurement context;
- VD: Some observables are *value indefinite*.

If we insist that *value definite* observables behave noncontextually, then only guaranteed that *some* observables are value indefinite

How Much Value Indefiniteness Is Reasonable?

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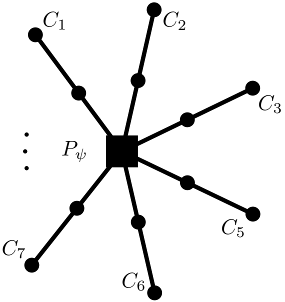

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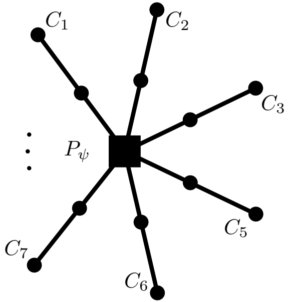
  - *One* direction of eigenvalue-eigenstate link.
- Intuitively, expect everything outside this 'star' to be value indefinite.



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  - VD″: An observable P assigned 1, and a *non-compatible* observable P' value definite.

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  - *One* direction of eigenvalue-eigenstate link.
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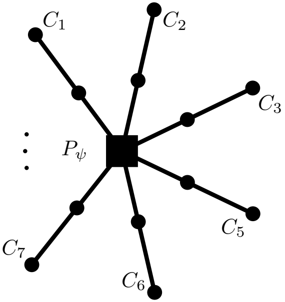
- Need to localise all the assumptions if we wish to go further formally.



Generalising the Formal Framework

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Admissibility of v

A value assignment function v is admissible if for every context $C \subset \mathcal{O}$:

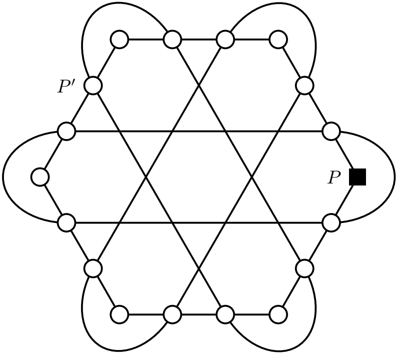
(a) if there exists a $P \in C$ with $v(P) = 1$, then $v(P') = 0$ for all $P' \in C \setminus \{P\}$;

(b) if there exists a $P \in C$ with $v(P') = 0$ for all $P' \in C \setminus \{P\}$, then $v(P) = 1$.

Failure of Existing Greechie Diagrams

Admissibility provides a way of deducing the value definiteness of observables.

Does there exist a set of observables $\mathcal{O}$ such that there is no admissible value assignment function with two non-compatible observables $P, P' \in \mathcal{O}$ and $v(P) = 1$ and P' value definite?



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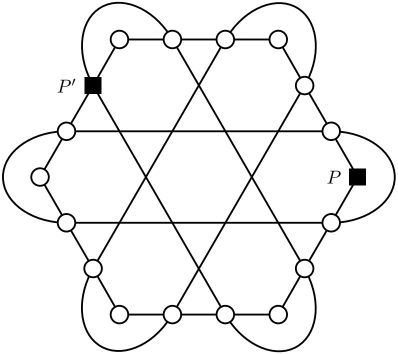

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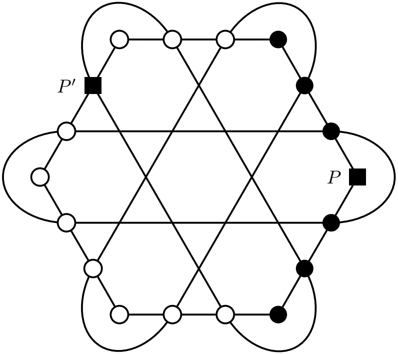

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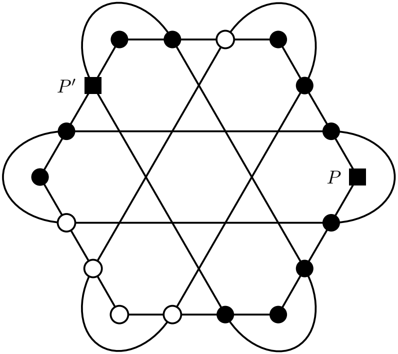

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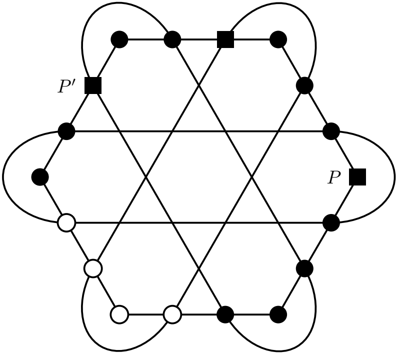

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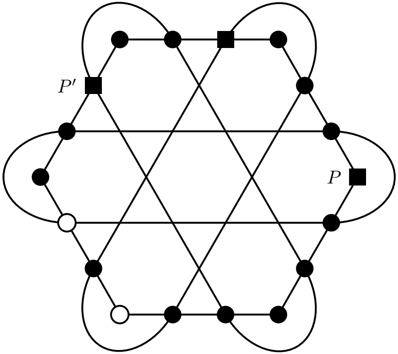

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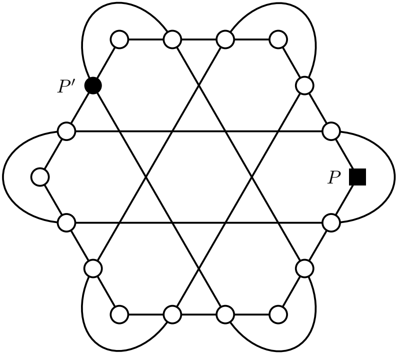

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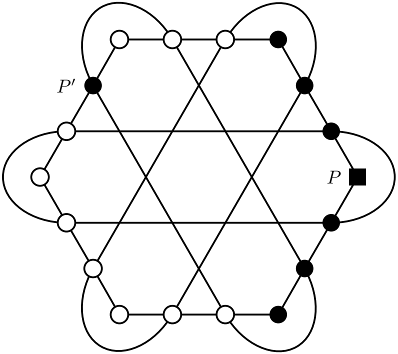
- Need more careful interlinking of observables to obtain a contradiction.
- To prove for all P' not compatible with P we either
  - (a) Need to consider $\mathcal{O}$ as the set of all projectors on $\mathbb{C}^n$;
  - (b) Give a procedure to find $\mathcal{O} = \mathcal{O}(P')$ for a given P'.



Localised Value Indefiniteness: A Theorem

Theorem

Let $n \geq 3$. If an observable P on $\mathbb{C}^n$ is assigned the value 1, then no other incompatible observable can be consistently assigned a definite value at all – i.e., is value indefinite.

Localised Value Indefiniteness: A Theorem

Theorem

Let $n \geq 3$ and $|\psi\rangle, |\phi\rangle \in \mathbb{C}^n$ be states such that $0 < |\langle\psi|\phi\rangle| < 1$. Then there is a finite set of observables $\mathcal{O}$ containing P_ψ and P_ϕ for which there is no admissible value assignment function on $\mathcal{O}$ such that $v(P_\psi) = 1$ and P_ϕ is value definite.

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We prove in 3 steps:

1. We first prove the explicit case that $|\langle\psi|\phi\rangle| = \frac{1}{\sqrt{2}}$.
2. We prove a reduction for $0 < |\langle\psi|\phi\rangle| < \frac{1}{\sqrt{2}}$ to the first case.
3. We prove a reduction for the last case of $\frac{1}{\sqrt{2}} < |\langle\psi|\phi\rangle| < 1$ case.

Proof

Interpretation and Conclusion

To interpret physically, need to connect P_ψ to the state of a system

Eigenstate value definiteness

If a system S is in a state $|\psi\rangle$, then $v_S(P_\psi) = 1$ for the value assignment function v_S modelling the system.

Can then conclude:

- For any state $|\psi\rangle$ can locate precisely the value indefinite observables
- Almost all observables are value indefinite (measure-one set)

General Interpretation

If a system is in a state $|\psi\rangle$, then the result of measuring an observable A is indeterministic unless $|\psi\rangle$ is an eigenstate of A.

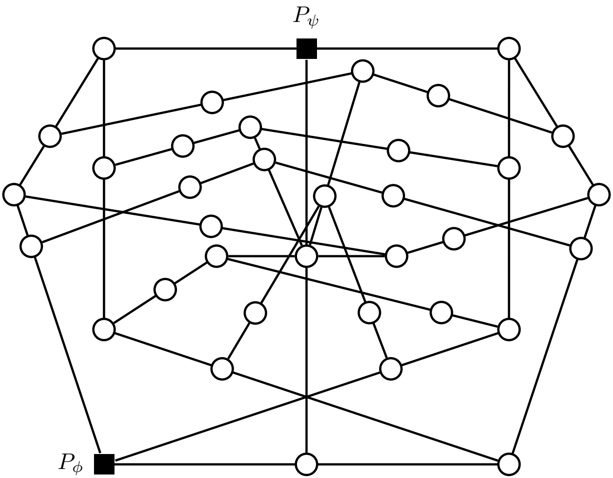
Thank You!

- A. A. Abbott, C. S. Calude & K. Svozil. *A variant of the Kochen-Specker theorem localising value indefiniteness*, arXiv:1503.01985, 2015.
- S. Kochen & E. Specker. *The problem of hidden variables in quantum mechanics*, Journal of Mathematics and Mechanics, 17:59–87, 1967 .
- A. Cabello et al. *Bell-Kochen-Specker Theorem: A proof with 18 vectors*, Phys. Lett. A, 212:183–187, 1996.
- A. A. Abbott, C. S. Calude, J. Conder & K. Svozil. *Strong Kochen-Specker theorem and incomputability of quantum randomness*. PRA, 86(062109), 2012

Explicit Case (Back)

The Greechie diagram below is realisable for $|\langle\psi|\phi\rangle| = \frac{1}{\sqrt{2}}$.

Assuming $v(P_\psi) = v(P_\phi) = 1$, we derive a contradiction:



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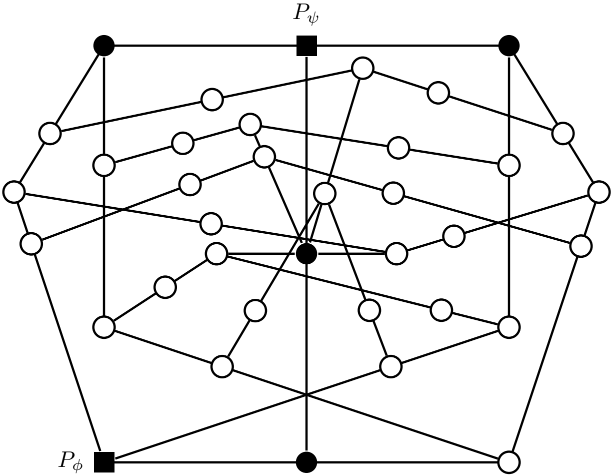

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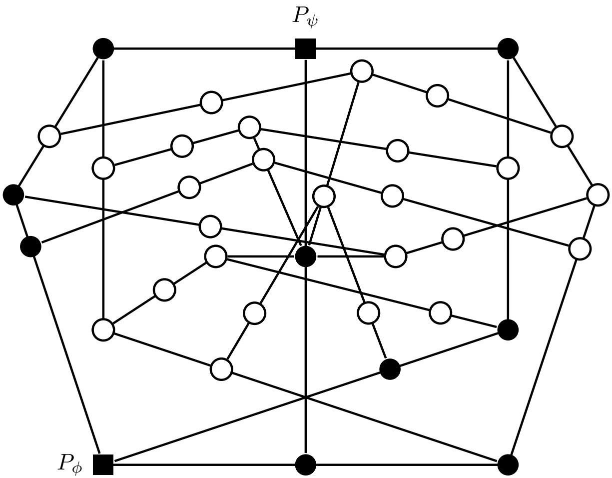

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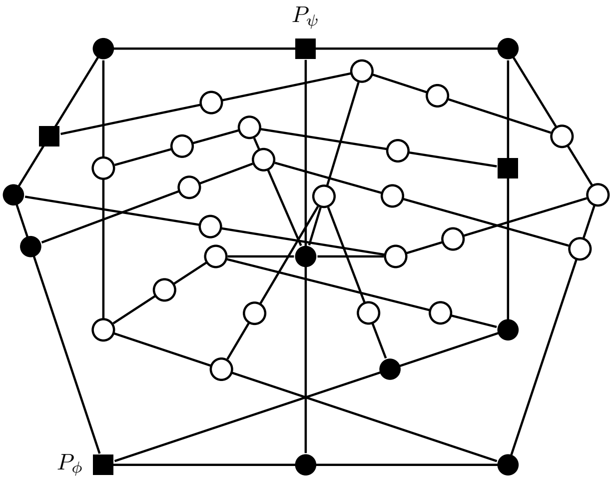

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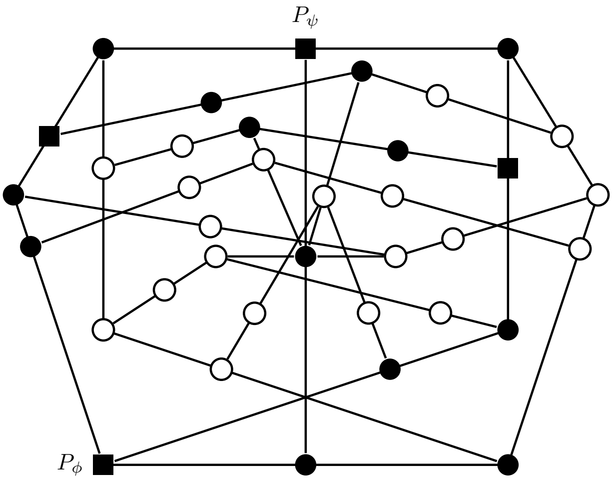

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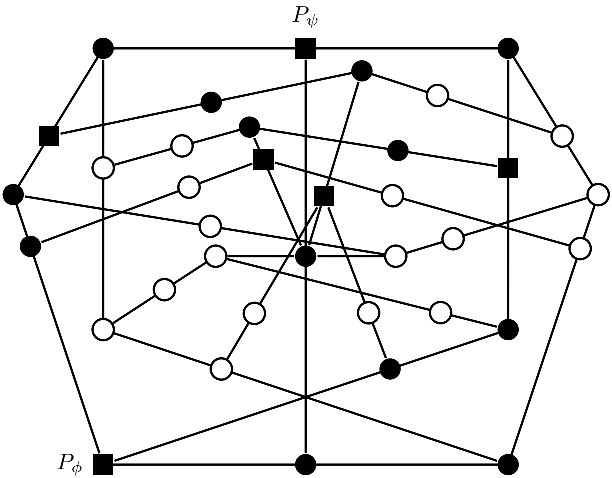

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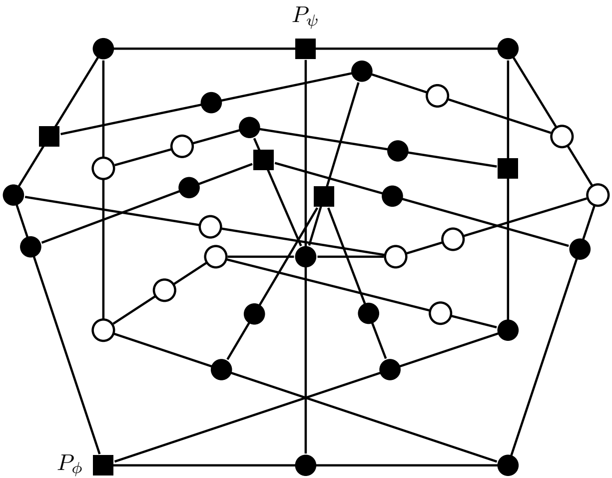

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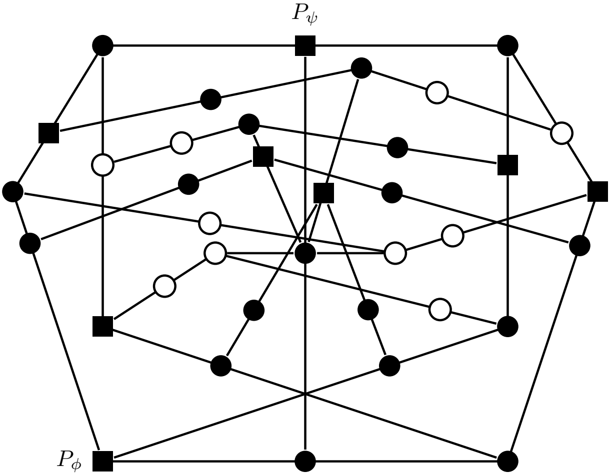

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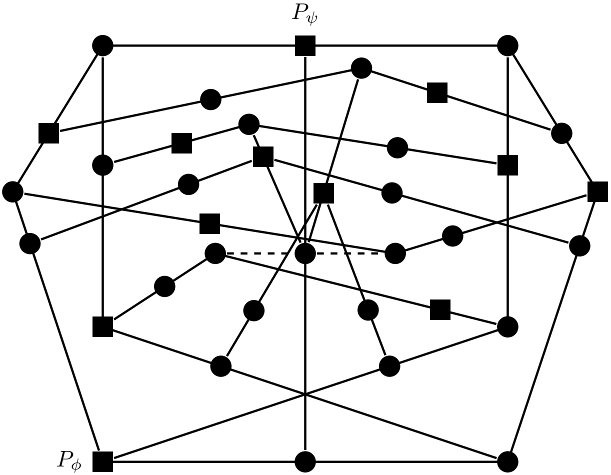

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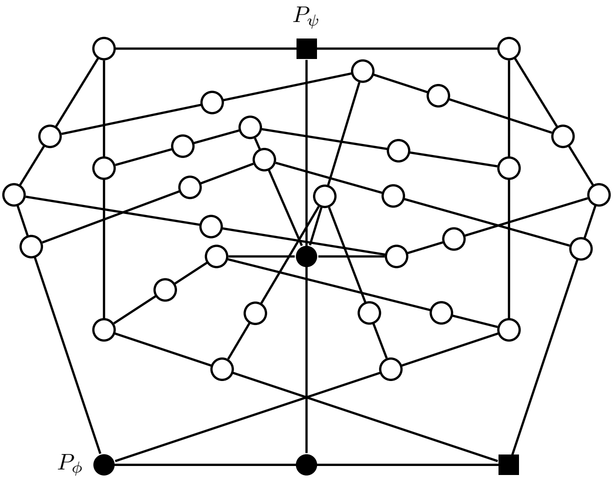

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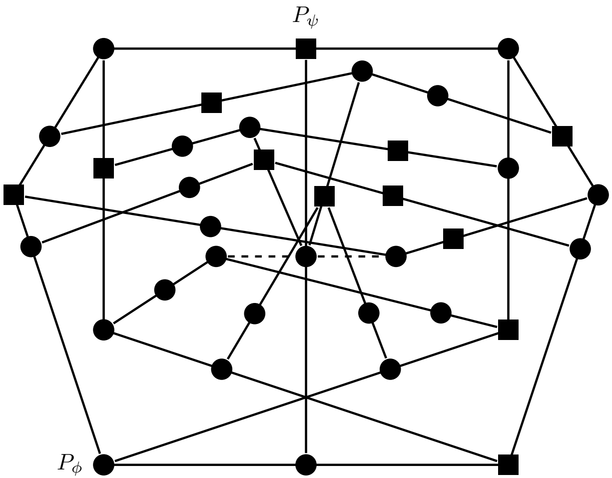
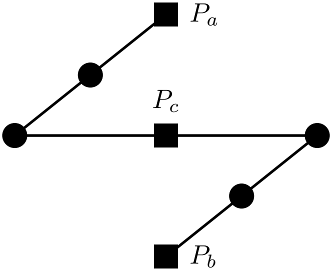


First Reduction: Contraction (Back)

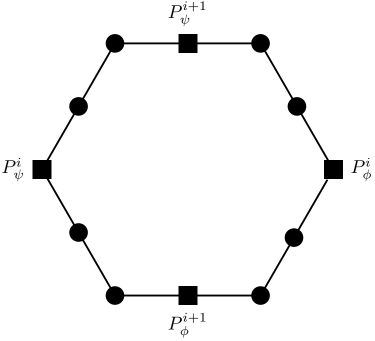
If $v(P_\psi) = v(P_\phi) = 1$ and $0 < |\langle\psi|\phi\rangle| < \frac{1}{\sqrt{2}}$, then we can find a $|\phi'\rangle$ with $\langle\psi|\phi'\rangle = \frac{1}{\sqrt{2}}$ and $v(P_{\phi'}) = 1$ under any admissible v.

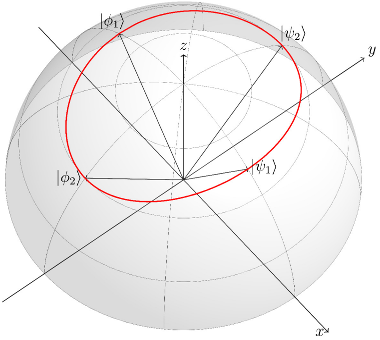


The above diagram is realisable for $|\langle a|b\rangle| < |\langle a|c\rangle| < 1$.

Second Reduction: Expansion (Back)

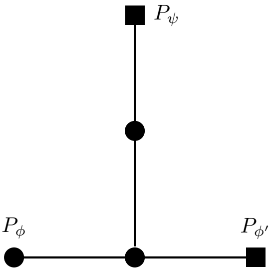
If $v(P_\psi) = v(P_\phi) = 1$ and $\frac{1}{\sqrt{2}} < |\langle\psi|\phi\rangle| < 1$, then we can find a finite sequence of states $|\psi_1\rangle, |\phi_1\rangle; \cdots, |\psi_n\rangle, |\phi_n\rangle$ such that for all i $v(P^i_\psi) = v(P^i_\phi) = 1$ and $\langle\psi_n|\phi_n\rangle = \frac{1}{\sqrt{2}}$ under any admissible v.





Completing the Proof (Back)

For the reductions we assumed $v(P_\phi) = 1$. If $v(P_\phi) = 0$, we can easily find $|\phi'\rangle$ with $v(P_{\phi'}) = 1$ and apply the reasoning above.



As a consequence, the set of value indefinite observables has measure 1: almost all observables are value indefinite.