Locating Value Indefiniteness with a Variant of the Kochen-Specker Theorem

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Joint work with C. S. Calude and K. Svozil
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Indeterminism and Quantum Randomness

- A random process or event is one that is unpredictable for any observer.
- Quantum randomness is generally reduced to the indeterminism of quantum measurements.

Born rule

Probability of obtaining 1 when measuring $P_\phi = |\phi\rangle\langle\phi|$ on a state $|\psi\rangle$ is $\Pr(\phi \mid \psi) = \langle\psi| P_\phi |\psi\rangle$.

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- Why should we interpret the Born rule as an objective probability distribution?
- Born: *"I myself am inclined to give up determinism in the world of atoms."*
- Bell's theorem and the Kochen-Specker theorem offer better evidence against determinism

Bell and Kochen-Specker Theorems

Bell's theorem

- The outcomes of certain measurements on certain states cannot be explained by local hidden variable theories.

Kochen-Specker theorem

- "Quantum contextuality"

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We still like to believe that all nontrivial measurements are indeterministic.

Eigenvalue-Eigenstate link

A system in a state $|\psi\rangle$ has a definite property of an observable A if and only if $|\psi\rangle$ is an eigenstate of A.

The Kochen-Specker Theorem

A *context* in $\mathbb{C}^n$ is a set of n compatible (commuting) observables.

In $n \geq 3$ Hilbert space there is a finite set of (projection) observables $\mathcal{O}$ such that

the following three are in contradiction:

1. Every observable is assigned a definite value of 0 or 1;
2. These definite values are noncontextual;
3. Exactly one observable in each context is assigned the value 1.

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there is no value assignment function $v : \mathcal{O} \to \{0, 1\}$ with:

1. v is total; i.e., $v(P)$ is defined for all $P \in \mathcal{O}$;
2. v is a function of P only;
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The Kochen-Specker Theorem cont.

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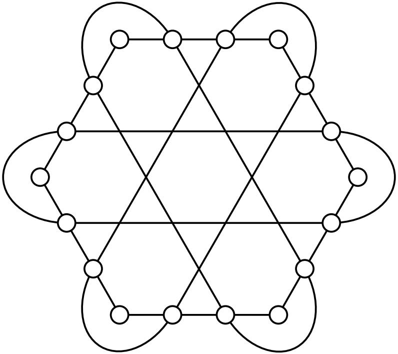

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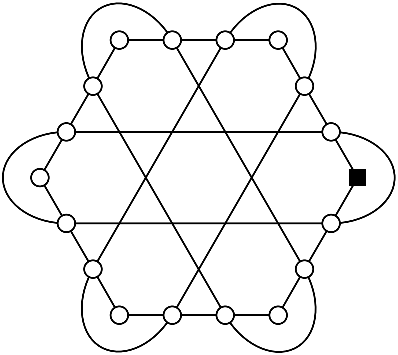

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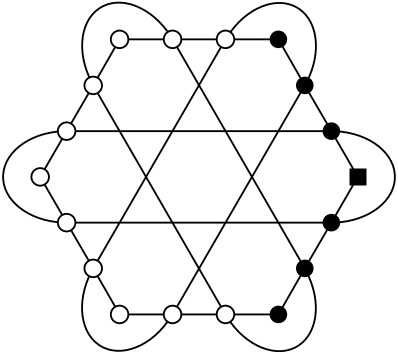

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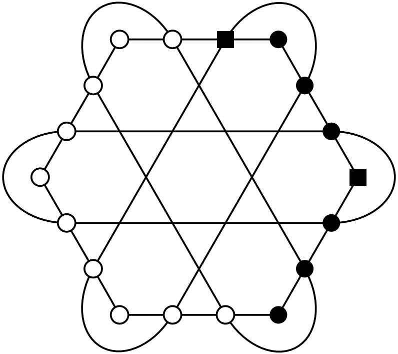

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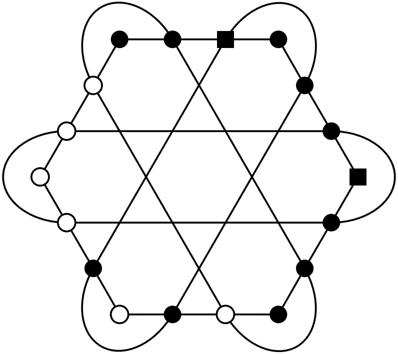

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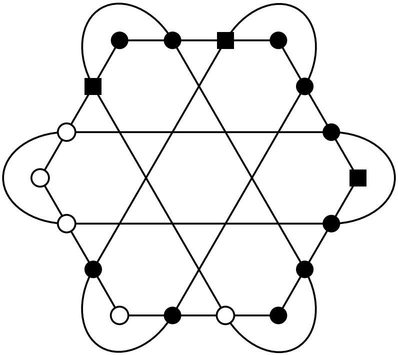

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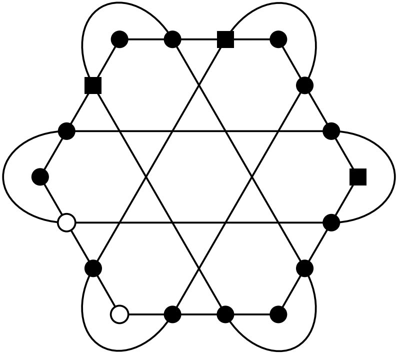

The Extent of Value Indefiniteness

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Either, we reject:

- QM: But then we depart from quantum theory;
- NC: Definite values depends on measurement context;
- VD: Some observables are *value indefinite*.

A value assignment function represents the measurement of an observable. Hence if we insist that *value definite* observables behave noncontextually, then *some* quantum measurements are indeterministic.

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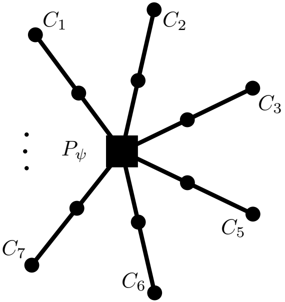

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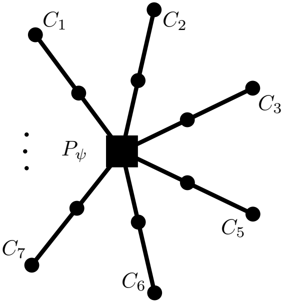
- If system is in state $|\psi\rangle$, reasonable to expect $v(P_\psi) = 1$.
  - *One* direction of eigenvalue-eigenstate link.
- Intuitively, expect everything outside this 'star' to be value indefinite.



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  - VD″: An observable P assigned 1, and a *non-compatible* observable P' value definite.

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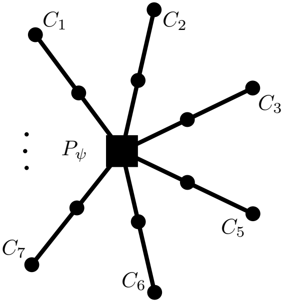
  - *One* direction of eigenvalue-eigenstate link.
- Intuitively, expect everything outside this 'star' to be value indefinite.
- Need to localise all the assumptions if we wish to go further formally.



Generalising the Formal Framework

- Consider a value assignment function $v : \mathcal{O} \rightarrow \{0, 1\}$ as a representation of the system, rather than a HVT
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  - Value indefinite observables are considered contextual.
- QM: How to handle condition that for all C, $\sum_{P \in C} v(P) = 1$ if $v(P)$ is undefined?

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Admissibility of v

A value assignment function v is admissible if for every context $C \subset \mathcal{O}$:

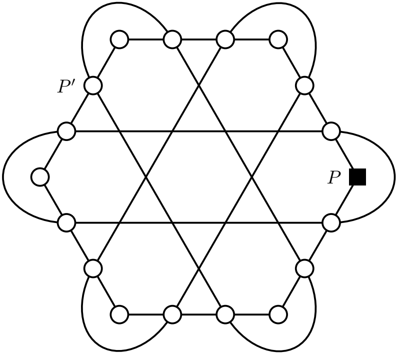
(a) if there exists a $P \in C$ with $v(P) = 1$, then $v(P') = 0$ for all $P' \in C \setminus \{P\}$;

(b) if there exists a $P \in C$ with $v(P') = 0$ for all $P' \in C \setminus \{P\}$, then $v(P) = 1$.

Failure of Existing Greechie Diagrams

Admissibility provides a way of deducing the value definiteness of observables.

Does there exist a set of observables $\mathcal{O}$ such that there is no admissible value assignment function with two non-compatible observables $P, P' \in \mathcal{O}$ and $v(P) = 1$ and P' value definite?



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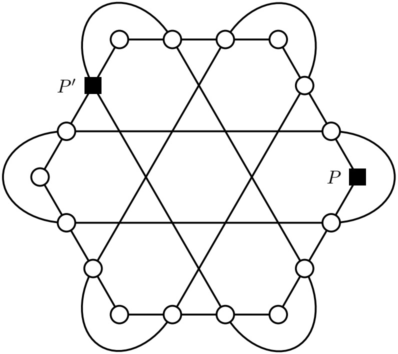

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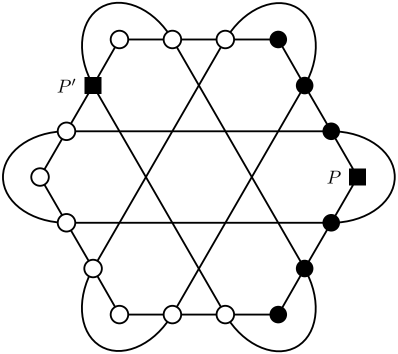

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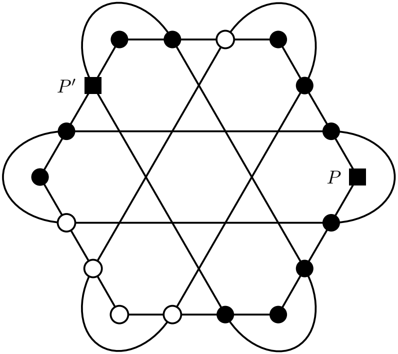

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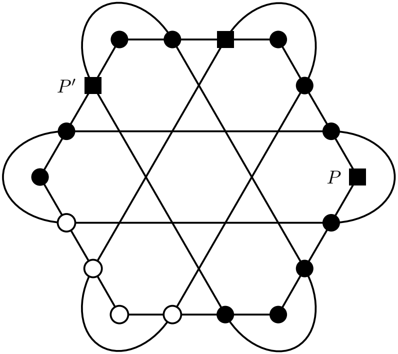

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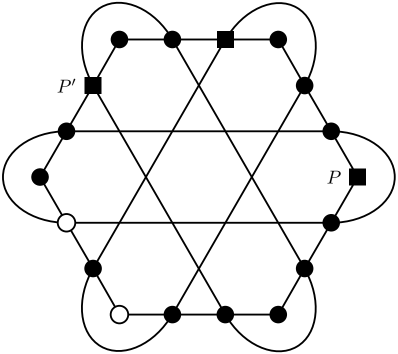

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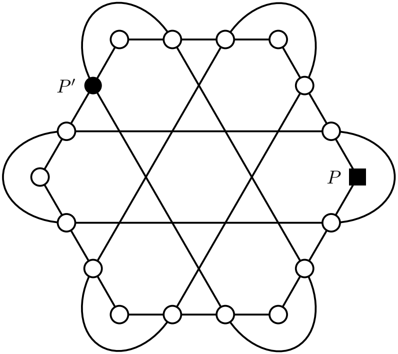

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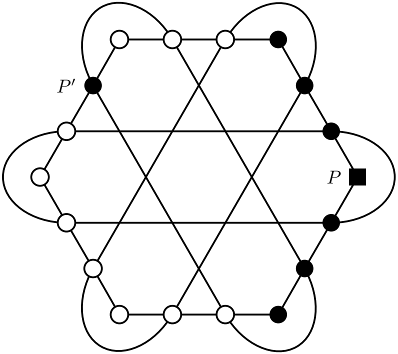
- Need more careful interlinking of observables to obtain a contradiction.
- To prove for all P' not compatible with P we either
  - (a) Need to consider $\mathcal{O}$ as the set off all projectors on $\mathbb{C}^n$;
  - (b) Give a procedure to find $\mathcal{O} = \mathcal{O}(P')$ for a given P'.



Localised Value Indefiniteness: A Theorem

Theorem

Let $n \geq 3$. If an observable P on $\mathbb{C}^n$ is assigned the value 1, then no other incompatible observable can be consistently assigned a definite value at all – i.e., is value indefinite.

Localised Value Indefiniteness: A Theorem

Theorem

Let $n \geq 3$ *and* $|\psi\rangle, |\phi\rangle \in \mathbb{C}^n$ *be states such that* $0 < |\langle\psi|\phi\rangle| < 1$. *Then there is a finite set of observables* $\mathcal{O}$ *containing* P_ψ *and* P_ϕ *for which there is no admissible value assignment function on* $\mathcal{O}$ *such that* $v(P_\psi) = 1$ *and* P_ϕ *is value definite.*

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We prove in 3 steps:

1. We first prove the explicit case that $|\langle\psi|\phi\rangle| = \frac{1}{\sqrt{2}}$.
2. We prove a reduction for $0 < |\langle\psi|\phi\rangle| < \frac{1}{\sqrt{2}}$ to the first case.
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Skip proof

Explicit Case

The Greechie diagram below is realisable for $|\langle\psi|\phi\rangle| = \frac{1}{\sqrt{2}}$.

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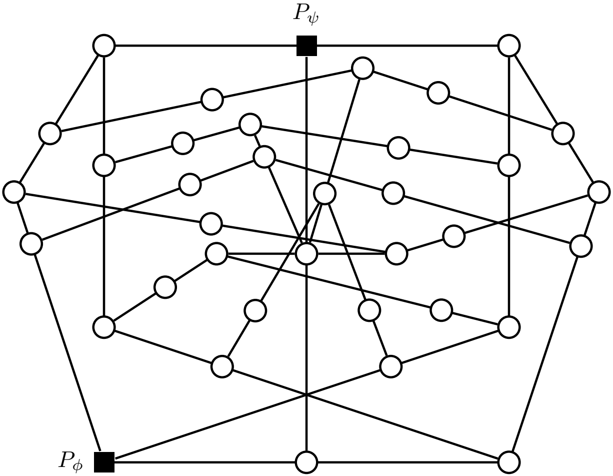

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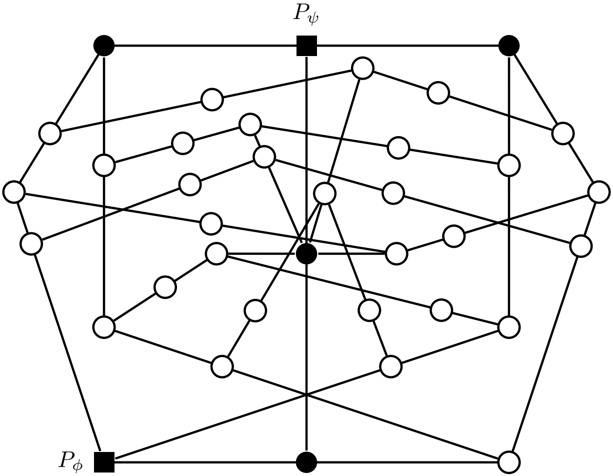

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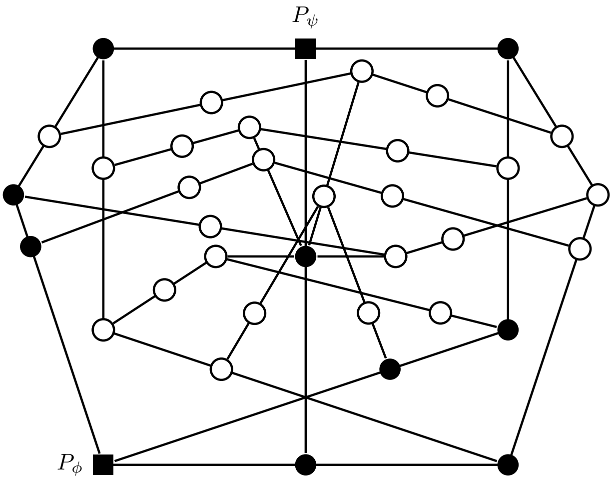

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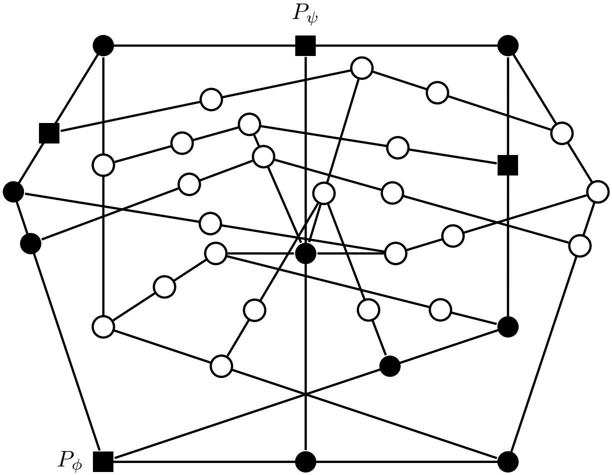

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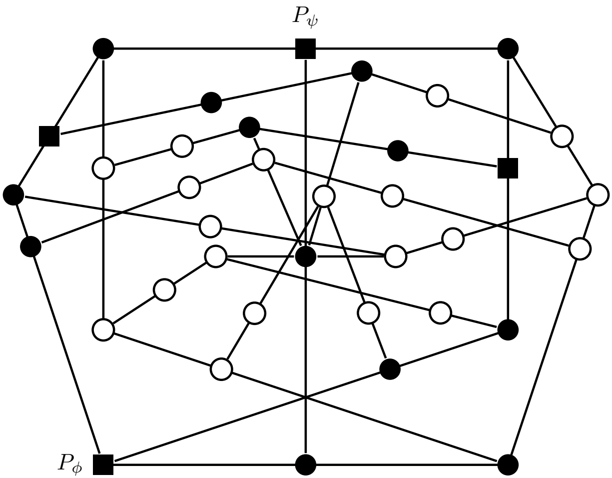

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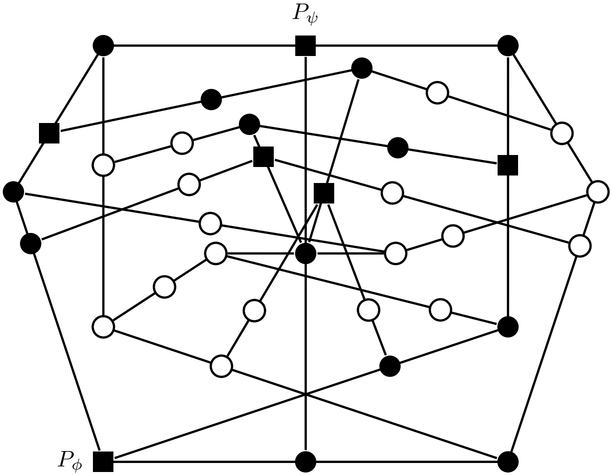

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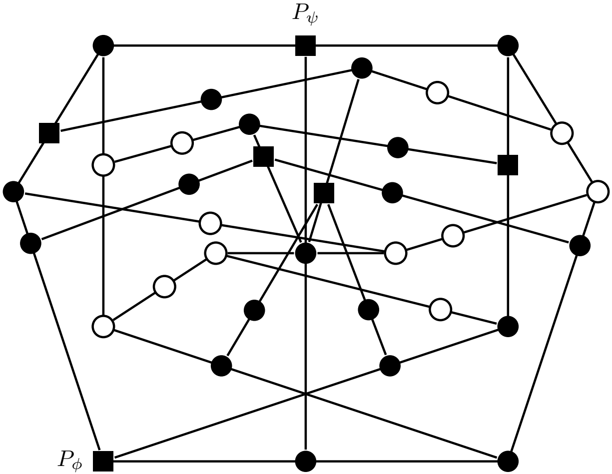

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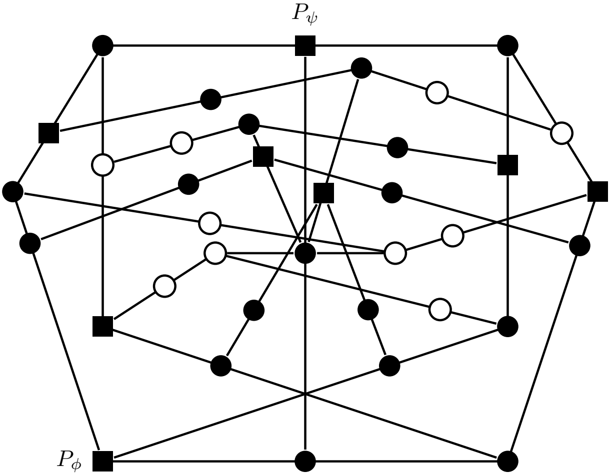

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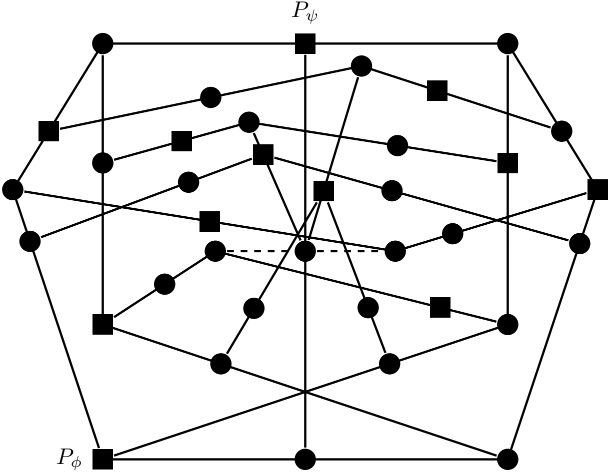

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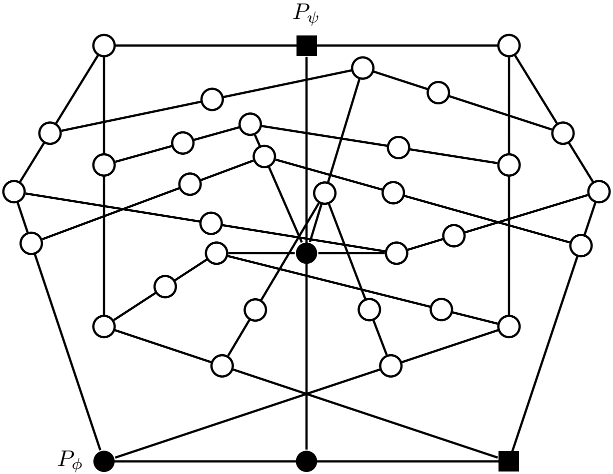

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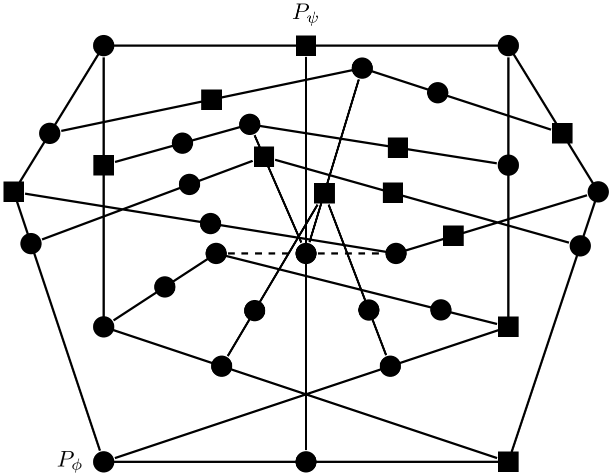

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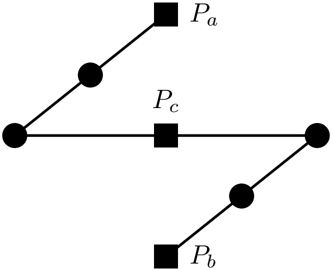
First Reduction: Contraction

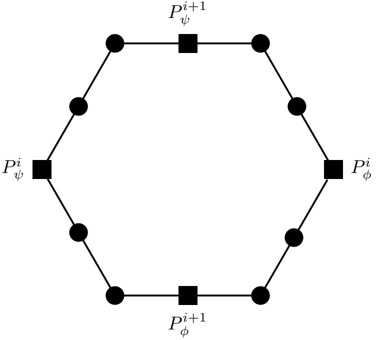
If $v(P_\psi) = v(P_\phi) = 1$ and $0 < |\langle\psi|\phi\rangle| < \frac{1}{\sqrt{2}}$, then we can find a $|\phi'\rangle$ with $\langle\psi|\phi'\rangle = \frac{1}{\sqrt{2}}$ and $v(P_{\phi'}) = 1$ under any admissible v.

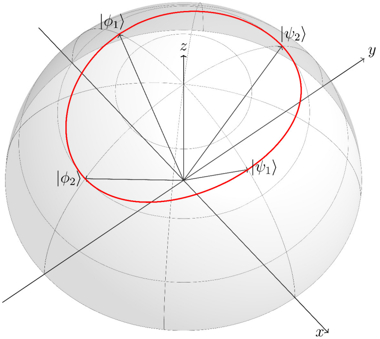


The above diagram is realisable for $|\langle a|b\rangle| < |\langle a|c\rangle| < 1$.

Second Reduction: Expansion

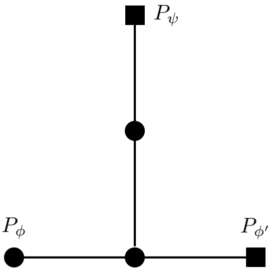
If $v(P_\psi) = v(P_\phi) = 1$ and $\frac{1}{\sqrt{2}} < |\langle\psi|\phi\rangle| < 1$, then we can find a finite sequence of states $|\psi_1\rangle, |\phi_1\rangle; \cdots, |\psi_n\rangle, |\phi_n\rangle$ such that for all i $v(P^i_\psi) = v(P^i_\phi) = 1$ and $\langle\psi_n|\phi_n\rangle = \frac{1}{\sqrt{2}}$ under any admissible v.





Completing the Proof

For the reductions we assumed $v(P_\phi) = 1$. If $v(P_\phi) = 0$, we can easily find $|\phi'\rangle$ with $v(P_{\phi'}) = 1$ and apply the reasoning above.



As a consequence, the set of value indefinite observables has measure 1: almost all observables are value indefinite.

A Physical Interpretation

This result is purely mathematical. How should we interpret it physically?

> *EPR: "If, without in any way disturbing a system, we can predict with certainty the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity."*

Eigenstate value definiteness

If a system is in a state $|\psi\rangle$, then $v(P_\psi) = 1$ for any *faithful* value assignment function v.

Interpretation

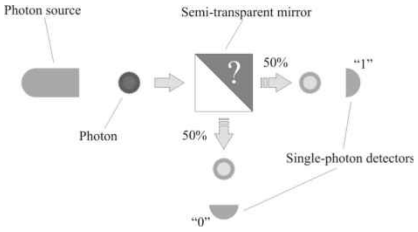
If a system is in a state $|\psi\rangle$, then the result of measuring an observable A is indeterministic unless $|\psi\rangle$ is an eigenstate of A.

Conclusion: Value Indefiniteness and Randomness

We assume *one* direction of the eigenvalue-eigenstate link, but *derive* the other direction.

The Kochen-Specker theorem shows that quantum-mechanics is indeterministic. This theorem shows the *extent* of this indeterminism and tells us precisely which observables are value indefinite.

- Subject to noncontextuality assumption
- Doesn't hold in two-dimensional Hilbert space.



References

- A. A. Abbott, C. S. Calude & K. Svozil. *A variant of the Kochen-Specker theorem localising value indefiniteness*, arXiv:1503.01985, 2015.
- S. Kochen & E. Specker. *The problem of hidden variables in quantum mechanics*, Journal of Mathematics and Mechanics, 17:59–87, 1967 .
- A. Cabello et al. *Bell-Kochen-Specker Theorem: A proof with 18 vectors*, Phys. Lett. A, 212:183–187, 1996.
- A. A. Abbott, C. S. Calude & K. Svozil. *On the unpredictability of individual quantum measurement outcomes*, arXiv:1403.2738, 2014.